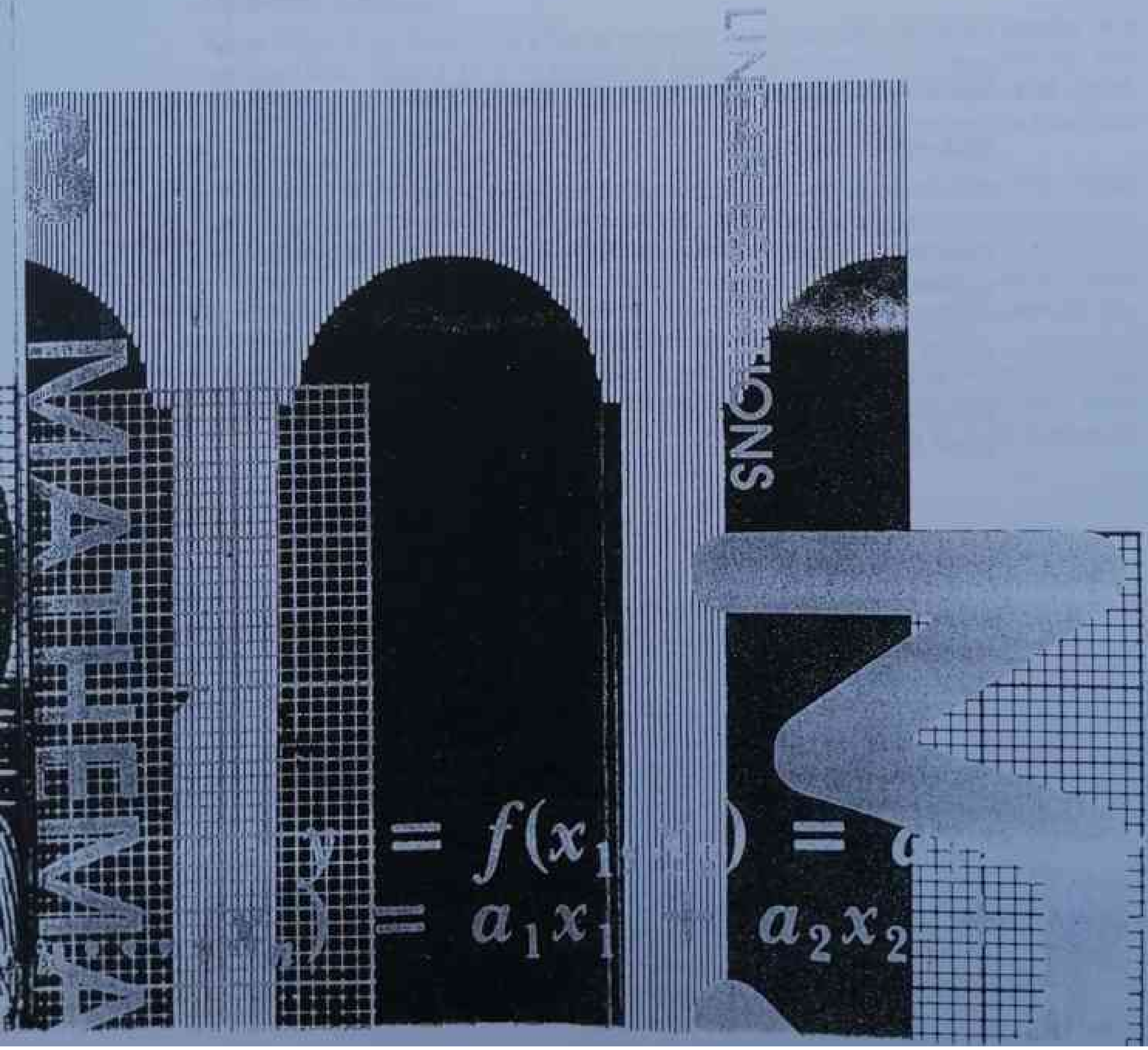
CHAPTER 3

SYSTEMS OF LINEAR EQUATIONS

CHAPTER OBJECTIVES

- Provide an understanding of the nature of systems of equations and their graphical representation (where appropriate)
- Provide an understanding of the different solution set possibilities for systems of equations
- Provide an appreciation of the graphical interpretation of solution sets
- Present procedures for determining solution sets for systems of equations
- Illustrate some applications of systems of linear equations



MOTIVATING SCENARIO: Emergency Airlift (continued)

Example 22 in Chap. 2 (page 73) was concerned with airlifting emergency supplies into a South American city. From Example 22 we know that the volume capacity of the plane equals 6,000 cubic feet. Another consideration is that the weight capacity of the plane is 40,000 pounds. In addition, the amount of money available for the purchase of supplies totals $150,000. Initial reports indicate that the most needed item is water. To respond to this need, Red Cross officials have specified that the number of containers of water shipped should be twice the combined number shipped of blood and medical supply kits. *Red Cross officials want to determine if there is some combination of the four items which will fill the plane to its weight and volume capacities, expend the full budget of $150,000, and satisfy the requirement regarding the shipment of water.* [Example 16]

In business, economics, or social science applications, we sometimes are interested in determining whether there are values of variables which satisfy several attributes. It may be that each attribute can be represented by using an equation, expressed in terms of the different variables. Together, the set of equations represents all of the attributes of interest. In this chapter, we will be concerned with the process used to determine whether there are values of variables which jointly satisfy a set of equations. For example, in the Motivating Scenario, we will see whether there are quantities of the four items which satisfy the attributes of weight capacity, volume capacity, budget, and water requirements.

3.1 TWO-VARIABLE SYSTEMS OF EQUATIONS

Systems of Equations

A ***system of equations*** is a set consisting of more than one equation. One way to characterize a system of equations is by its ***dimensions.*** If a system of equations consists of m equations and n variables, we say that this system is an ***"m by n" system***, or that it has dimensions $m \times n$. A system of equations involving 2 equations and 2 variables is described as having dimensions 2×2. A system consisting of 15 equations and 10 variables is said to be a (15×10) system.

In solving systems of equations, we are interested in identifying values of the variables that satisfy all equations in the system simultaneously. For example, given the two equations

$$5x + 10y = 20$$
$$3x + 4y = 10$$

we may wish to identify any values of x and y which satisfy both equations at the same time. Using set notation, we would want to identify the ***solution set S,*** where

$$S = \{(x, y) \mid 5x + 10y = 20 \quad \text{and} \quad 3x + 4y = 10\}$$

As you will see in this chapter, the solution set S for a system of linear equations may be a ***null set***, a ***finite set***, or an ***infinite set***.*

There are quite a few solution procedures which may be used in solving systems of equations. In this chapter, we concentrate on two different procedures. Other procedures will be presented in Chap. 9.

We begin our discussion in this chapter with the simplest systems, two equations and two variables. Our discussions will emphasize both the graphical and algebraic aspects of each situation. These procedures will be extended later in the chapter to acquaint us with how larger systems of equations are handled. We will also discuss a variety of applications of systems of equations.

Graphical Analysis

From Chap. 2 we know that a linear equation involving two variables graphs as a straight line. Thus a (2×2) system of linear equations is represented by two straight lines in two dimensions. In solving for the values of the two variables which satisfy *both* equations, we are graphically trying to determine whether the two lines have any points in common.

For (2×2) systems of equations three different types of solution sets might exist. Figure 3.1 illustrates the three possibilities. In Fig. 3.1*a*, the two lines intersect. The *coordinates* of the point of intersection (x_1, y_1) represent the solution for the system of equations, i.e., the pair of values for x and y which satisfy *both* equations. When there is just one pair of values for the variables which satisfy the system of equations, the system is said to have a ***unique solution***.

In Fig. 3.1*b*, the two lines are parallel to each other. You should remember from Chap. 2 that parallel lines have the same slope; and provided that they have different y intercepts, the lines have no points in common. If a (2×2) system of

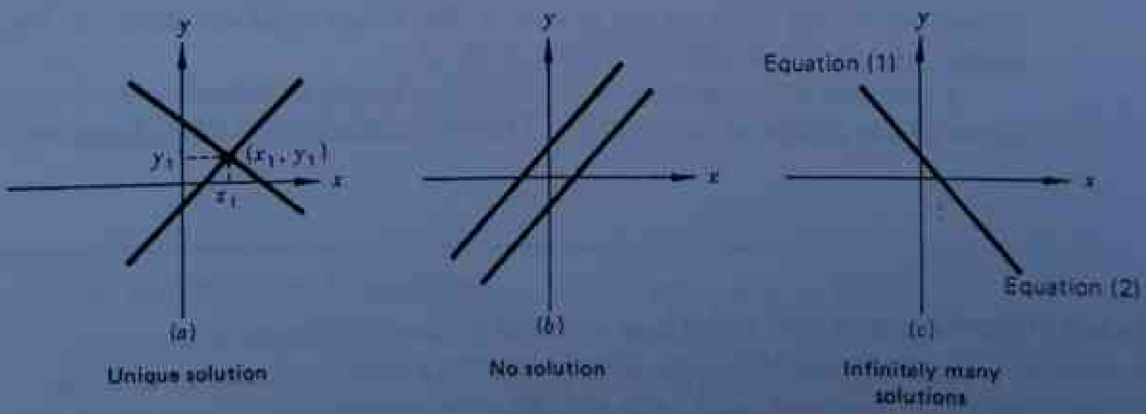


Figure 3.1 Solution set possibilities for a (2×2) system of equations.

* A null set contains no elements (it is empty), a finite set consists of a limited number of elements, and an infinite set consists of an infinite number of elements.

Figure

equations has these characteristics, the system is said to have ***no solution.*** That is, there are no values for the variables which satisfy both equations. The equations in such a system are said to be ***inconsistent.***

The final possibility for a (2 × 2) system is illustrated in Fig. 3.1c. In this case both equations graph as the same line, and they are considered to be ***equivalent equations.*** An infinite number of points are common to the two lines, and the system is said to have ***infinitely many solutions.*** Being represented by the same line implies that both lines have the same slope *and* the same *y* intercept. Two equations *can* look very different from each other and still be equivalent to one another. For example, the two equations

$$-6x + 12y = -24$$

and

$$1.5x - 3y = 6$$

are equivalent. Verify that the slope and the *y* intercept are the same for both.

Another way of summarizing the three cases illustrated in Fig. 3.1 is as follows.

SLOPE-INTERCEPT RELATIONSHIPS

Given a (2 × 2) system of linear equations (in slope-intercept form),

$$y = m_1x + k_1 \tag{3.1}$$

$$y = m_2x + k_2 \tag{3.2}$$

where m_1 and m_2 represent the respective slopes of the two lines and k_1 and k_2 represent the respective y intercepts.

I *There is a* unique solution *to the system if* $m_1 \neq m_2$.

II *There is* no solution *to the system if* $m_1 = m_2$ *but* $k_1 \neq k_2$.

III *There are* infinitely many solutions *if* $m_1 = m_2$ *and* $k_1 = k_2$.

Graphical Solutions

Graphical solution approaches are possible for two-variable systems of equations. However, you must be accurate in your graphics. The following example illustrates a graphical solution.

IPLE 1 Graphically determine the solution to the system of equations

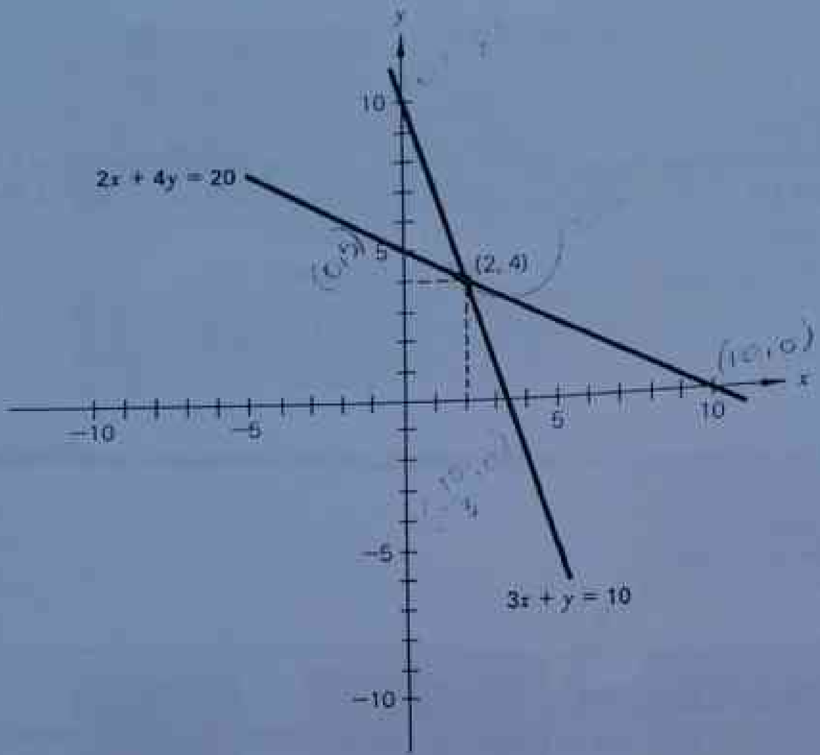
$$2x + 4y = 20 \tag{3.3}$$

$$3x + y = 10 \tag{3.4}$$

The x and y intercepts are, respectively, (10, 0) and (0, 5) for Eq. (3.3). Similarly, the intercepts for Eq. (3.4) are ($\frac{10}{3}$, 0) and (0, 10). When these are plotted in Fig. 3.2 and connected, the two lines appear to cross at (2, 4).

A problem with graphical solutions is that it may be difficult to read the precise coordinates of the points of intersection. This is especially true when the coordinates are not

Figure 3.2



integers. This is why algebraic solution procedures are preferred from the standpoint of identifying *exact* solutions. However, whether you use graphical or algebraic procedures, there is always a check on your answer: Substitute your answer into the original equations to see whether they are satisfied by the values. Substituting $x = 2$ and $y = 4$ into Eqs. (3.3) and (3.4), we get

$$2(2) + 4(4) = 20$$

or

$$20 = 20$$

and

$$3(2) + (4) = 10$$

or

$$10 = 10$$

Therefore our solution is correct.

□

The Elimination Procedure

One popular solution method for two- and three-variable systems is the ***elimination procedure.*** Given a (2×2) system of equations, the two equations, or multiples of the two equations, are added so as to *eliminate* one of the two variables. The resultant equation is stated in terms of the remaining variable. This equation can be solved for the remaining variable, the value of which can be substituted back into one of the original equations to solve for the value of the eliminated variable. The solution process is demonstrated in the following example, after which the procedure will be formalized.

EXAMPLE 2 Solve the system of equations in Example 1.

SOLUTION

The original system was

$$2x + 4y = 20 \tag{3.3}$$

$$3x + y = 10 \tag{3.4}$$

The objective of the elimination procedure is to eliminate one of the two variables by adding (multiples of) the equations. If we *multiply* Eq. (3.4) by -4 and *add* the resulting equation [Eq. (3.4*a*)] to Eq. (3.4), we get Eq. (3.5):

$$2x + 4y = 20 \tag{3.3}$$

$$[-4 \cdot \text{Eq. } (3.4)] \rightarrow \quad -12x - 4y = -40 \tag{3.4a}$$

$$-10x = -20 \tag{3.5}$$

Equation (3.5) contains the variable x only and can be solved for the value $x = 2$. Substituting this value for x into one of the original equations—let's select Eq. (3.3)—we find that

$$2(2) + 4y = 20$$

$$4y = 16$$

or

$$y = 4$$

Therefore the unique solution to the system, as we determined graphically, is $x = 2$ and $y = 4$.

☐

EXAM

PRACTICE EXERCISE

Verify that the solution is exactly the same if x is selected for elimination. To eliminate x, multiply Eqs. (3.3) and (3.4) by -3 and 2, respectively.

The elimination procedure can be generalized as follows for a (2×2) system of equations.

ELIMINATION PROCEDURE FOR (2 × 2) SYSTEMS

I *Select a variable to eliminate.*

II *Multiply (if necessary) the equations by constants so that the coefficients on the selected variable are the negatives of one another in the two equations, and add the two resulting equations.*

III (A) *If adding the equations results in a new equation having one variable, there is a* **unique solution** *to the system. Solve for the*

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value of the remaining variable, and substitute this value back into one of the original equations to determine the value of the variable that was originally eliminated.

(B) *If adding the equations results in the* **identity** $0 = 0$, *the two original equations are* **equivalent** *to each other and there are* **infinitely many solutions** *to the system.*

(C) *If adding the equations results in a* **false statement,** *say,* $0 = 5$, *the equations are* **inconsistent** *and there is* **no solution.** *See Fig. 3.3.*

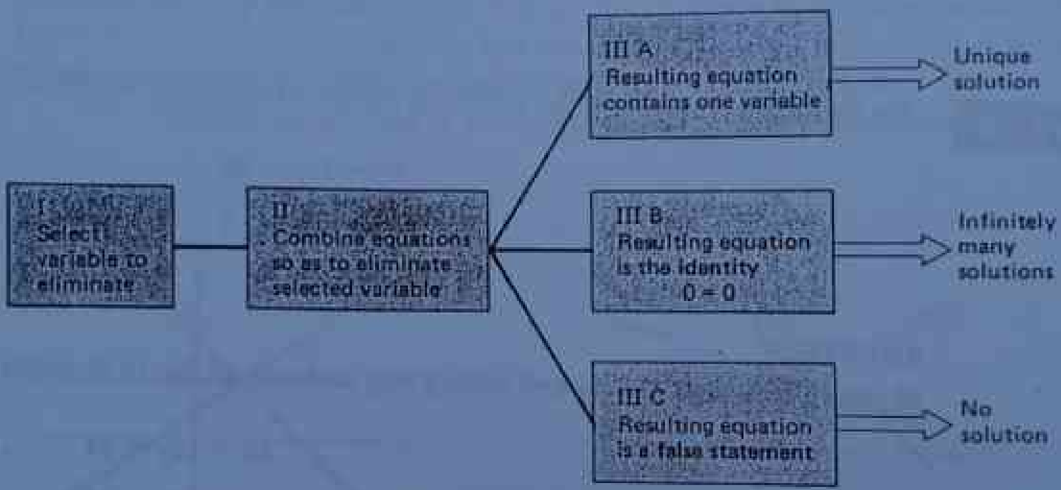


Figure 3.3 Elimination procedure for (2×2) systems.

EXAMPLE 3 **(Infinitely Many Solutions)** Solve the following system of equations by the elimination procedure.

$$3x - 2y = 6 \tag{3.6}$$

$$-15x + 10y = -30 \tag{3.7}$$

SOLUTION

Choosing x as a variable to eliminate, Eq. (3.6) is multiplied by 5 and added to Eq. (3.7).

$$[5 \cdot \text{Eq. } (3.7)] \rightarrow \quad 15x - 10y = 30 \tag{3.6a}$$

$$\underline{-15x + 10y = -30} \tag{3.7}$$

$$0 = 0$$

When Eqs. (3.6a) and (3.7) are added, both variables are eliminated on the left side of the equation and we are left with the identity $0 = 0$. From step IIB of the solution procedure we conclude that the two equations are equivalent and there are infinitely many solutions.

In order to specify sample members of the solution set, we could assume an arbitrary value for either x or y and substitute this value into one of the original equations, solving for the corresponding value of the other variable. For example, verify that if we let $y = 3$, substitution of this value into either Eq. (3.6) or (3.7) will result in the corresponding value $x = 4$. Thus, one member of the solution set is (4, 3). A more general way of specifying the solution set is to solve either of the original equations for one of the variables. The result is an equation which states the value of one variable in terms of the value of the second variable. To illustrate, if Eq. (3.6) is solved for x, the result is

$$3x = 2y + 6$$

or

$$x = \tfrac{2}{3}y + 2$$

Therefore, one way of generalizing the solution set is

y arbitrary
$x = \frac{2}{3}y + 2$

Very simply, this notation states that y may be assigned *any* real value and the corresponding value for x is obtained by substitution into the equation $x = \frac{2}{3}y + 2$. Alternatively, the solution set might be generalized by solving either of the original equations for y. Verify that the resulting generalization would have the form

x arbitrary
$y = \frac{3}{2}x - 3$

XAMPLE 4 **(No Solution Set)** Solve the following system of equations by the elimination procedure.

$$6x - 12y = 24 \quad (3.8)$$

$$-1.5x + 3y = 9 \quad (3.9)$$

SOLUTION

Multiplying Eq. (3.9) by 4 and adding this multiple to Eq. (3.8) yields

$$\begin{array}{rl} & 6x - 12y = 24 \quad (3.8) \\ [4 \cdot \text{Eq. } (3.9)] \rightarrow & \underline{-6x + 12y = 36} \quad (3.9a) \\ & 0x + 0y = 60 \end{array}$$

or

$$0 = 60$$

Since $0 = 60$ is a false statement, there is no solution to the system of equations. □

PRACTICE EXERCISE
Rewrite Eqs. (3.8) and (3.9) in slope-intercept form and confirm that they have the same slope but different y intercepts.

$(m \times 2)$ Systems, $m > 2$

When there are more than two equations ($m > 2$) involving two variables, each equation still graphs as a line in two dimensions. For example, Fig. 3.4 illustrates two (3×2) systems. In Fig. 3.4a the three lines all intersect at the same point, and there is a unique solution. In Fig. 3.4b there are points which are common to different pairs of lines, but there is no point common to all three, which means that there is no solution. A possible, but unlikely, situation is that the m equations are all equivalent to one another and all graph as the same line.

The solution procedure is relatively simple for these systems.

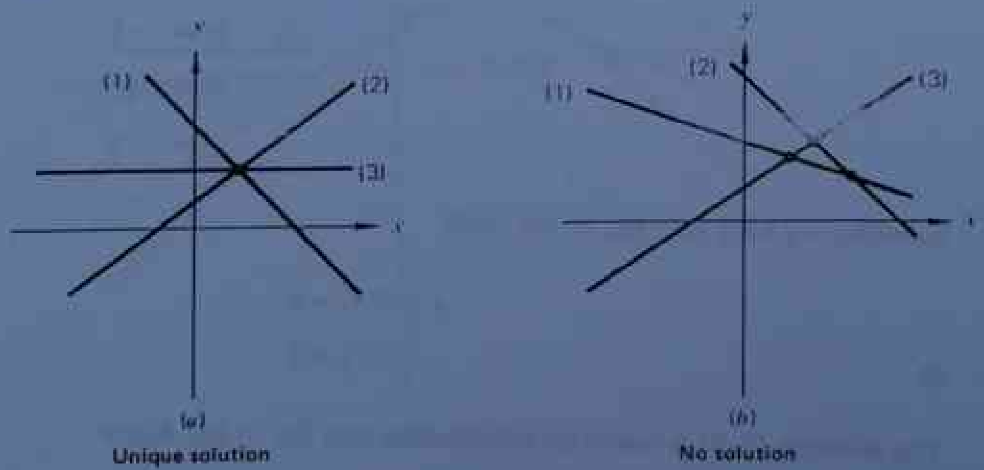


Figure 3.4 Solution possibilities for (3×2) systems.

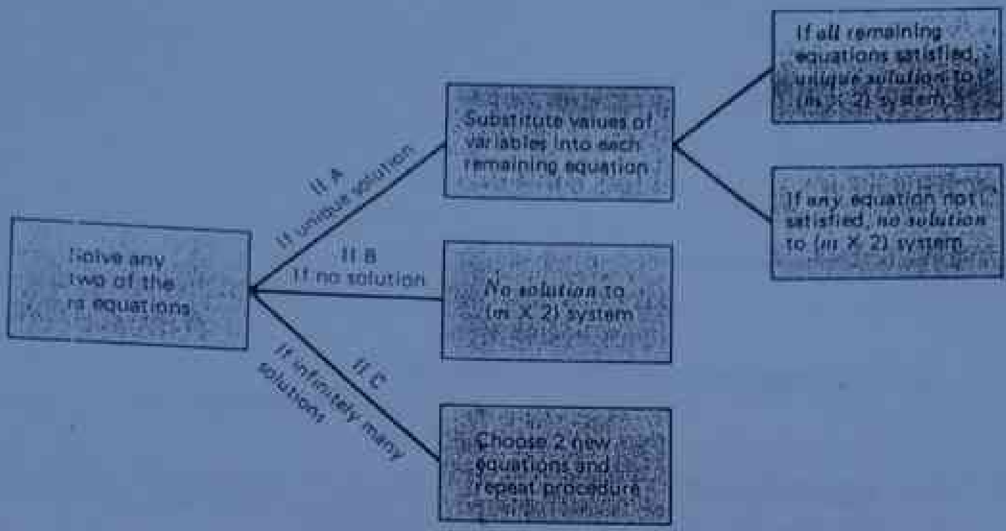


Figure 3.5 Elimination procedure for $(m \times 2)$ systems, $m > 2$.

EXAMPLE 5 Determine the solution set for the following system of equations:

$$x + 2y = 8 \tag{3.10}$$

$$2x - 3y = -5 \tag{3.11}$$

$$-5x + 6y = 8 \tag{3.12}$$

$$x + y = 7 \tag{3.13}$$

SOLUTION

The (2×2) system consisting of Eqs. (3.10) and (3.11) is solved by multiplying Eq. (3.10) by -2 and adding it to Eq. (3.11), or

$$\begin{array}{rr} [-2 \cdot \text{Eq. } (3.10)] \rightarrow & -2x - 4y = -16 \\ & \underline{2x - 3y = \;\; -5} \\ & -7y = -21 \\ & y = \quad 3 \end{array}$$

Substituting back into Eq. (3.10) yields

$$x + 2(3) = 8$$

or

$$x = 2$$

The solution (2, 3) is tested by substituting into Eq. (3.12). Since

$$-5(2) + 6(3) = 8$$

or

$$8 = 8$$

the point (2, 3) satisfies the first three equations. Substituting into Eq. (3.13) gives

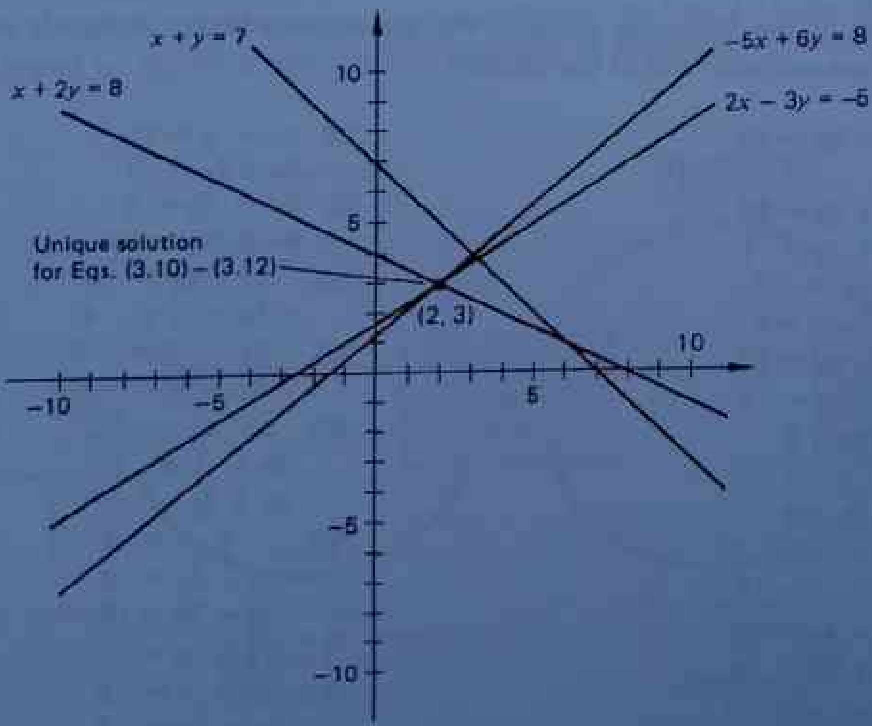
$$2 + 3 \neq 7$$

or

$$5 \neq 7$$

Since (2, 3) does not satisfy Eq. (3.13), there is no unique solution to the system of equations. Figure 3.6 illustrates the situation. Note that the lines representing Eqs. (3.10) — (3.12) intersect at the point (2, 3); however (2, 3) does not lie on the line representing Eq. (3.13). ☐

Figure 3.6
No solution for (4 × 2) system.



1 $5x + 5y = 0$
$x = -y$

2 $2x - 9y = 108$
$8x + 6y = 48$

3 $4x - 2y = 8$
$x + 2y = 12$

4 $3x - 9y = 24$
$-x + 3y = 0$

5 $x - 3y = 8$
$-4x + 12y = -24$

6 $4x - 2y = 36$
$-2x + y = 20$

7 $-3x = y + 2$
$9x + 3y = -6$

8 $x + y = 20$
$2x - y = 12$

9 $-x = -y$
$3x + 3y = 0$

10 $4x - y = 10$
$2x + 3y = 18$

In Exercises 11–20, solve graphically and check your answer algebraically.

11 $2x - 3y = -13$
$4x + 2y = -2$

12 $3x + 2y = 8$
$x - y = 1$

13 $-x + 2y = -2$
$3x = 6y + 6$

14 $x - 2y = 0$
$-3x + 6y = 5$

15 $3x + 4y = 5$
$4x + y = -2$

16 $x - 2y = 4$
$-4x + 8y = -10$

17 $4x - 2y = 10$
$-2x + y = -5$

18 $x + y = 0$
$-2x + 3y = 10$

19 $-x + 3y = 2$
$4x - 12y = -8$

20 $-x + y = 0$
$2x + y = 9$

Figure 3.7
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(2 × 2) syster

Solve each of the following systems. For any system having infinitely many solutions, specify a generalized form of the solution.

21 $4x - 2y = 20$
$-2x = -y + 15$

22 $4x - y = 17$
$5x + 3y = 0$

23 $-2x + 5y = 20$
$4x + y = 4$

24 $6x - 8y = 4$
$6 + 12y = 9x$

25 $2x - y = 9$
$x + 3y = -6$

26 $2x + 4y = -8$
$-3x + 2y = 4$

27 $12x - 4y = 18$
$-4x + y = 6$

28 $2x - y = 4$
$-6x + 3y = -12$

29 $x - y = 2$
$2x + y = 1$
$7x - 5y = 6$

30 $x - 2y = -7$
$3x + y = 0$
$2x + 3y = 7$

31 $x + y = 3$
$2x - y = 12$
$x - 4y = 13$
$-2x + 5y = 0$

32 $x + y = 4$
$2x - 3y = 3$
$4x - 2y = 10$
$-x + 3y = 0$

33 $x - y = 1$
$x + 2y = -8$
$3x - 2y = 0$
$2x - 5y = 11$
$-4x + 3y = -1$

34 $x - y = 8$
$2x + y = 4$
$3x + 2y = 4$
$x + 2y = -4$
$5x - 2y = 20$

10 A coffee manufacturer is interested in blending five types of coffee beans into a final blend of 120,000 pounds of coffee. The five component beans cost \$2, \$3, \$4, \$2, and \$2 per pound, respectively. The budget for purchasing the five components is \$300,000. In blending the coffee, three restrictions have been stated: (1) the combination of components 1 and 2 should constitute exactly half of the final blend; (2) components 1 and 5 together should constitute exactly 25 percent of the final blend; and (3) the amount of component 4 used in the blend should be exactly three times the amount used of component 3.

(*a*) Formulate the system of equations which states all requirements in this blending problem.

(*b*) How many pounds of each component should be used in the final blend?

APPENDIX: ELIMINATION PROCEDURE FOR (3 × 3) SYSTEMS

The elimination procedure for (3 × 3) systems is similar to that for (2 × 2) systems. The aim is to start with the (3 × 3) system and to reduce this to an equivalent system having two variables and two equations. With one of the three variables eliminated, the same procedure as used for (2 × 2) systems is employed to eliminate a second variable, resulting in a (1 × 1) system. After you solve for the remaining variable, its value is substituted sequentially back through the (2 × 2) system and finally the (3 × 3) system to determine the values of the other two variables. Figure 3.13 illustrates the process schematically. In Fig. 3.13, x_1 is eliminated first, followed by x_2. This order is not required; it is simply illustrative.

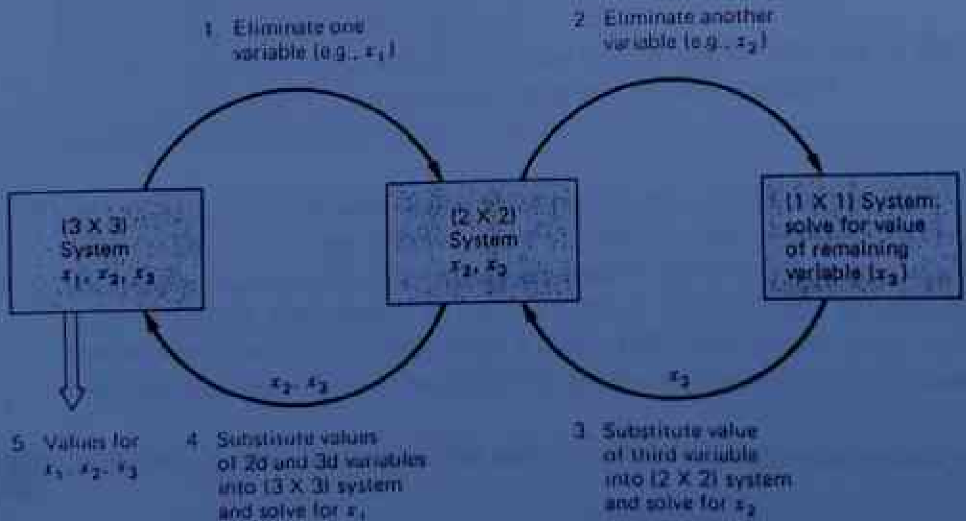

Figure 3.13 Elimination procedure for (3 × 3) systems.

The elimination procedure for a (3 × 3) system is as follows.

ELIMINATION PROCEDURE FOR 3 × 3 SYSTEMS

I *Add multiples of any two of the three equations in order to eliminate one of the three variables. The result should be an equation involving the other two variables.*

II *Repeat step I with another pair of the original equations, eliminating the same variable as in step I. This second pair of equations will include one of the two equations used in step I and the equation not used in step I.*

III *The results of steps I and II should be a (2 × 2) system. Use the procedure for (2 × 2) systems (page 90) to determine the values for the remaining two variables.*

IV *Substitute the values of these two variables into one of the original equations. Solve for the value of the third variable.*

If during any phase of the elimination procedure an identity results [see step IIIB of the (2 × 2) procedure], then the solution set contains an infinite number of elements. An exception to this is the case in which step I results in an identity and step II a false statement, or contradiction. What is the graphical implication of these two results? *If at any stage a false statement results [step IIIC of the (2 × 2) procedure], then there is no solution to the original system of equations.*

EXAMPLE 20

Unique Solution Determine the solution set for the following system of equations.

$$x_1 + x_2 + x_3 = 6 \tag{3.22}$$

$$2x_1 - x_2 + 3x_3 = 4 \tag{3.23}$$

$$4x_1 + 5x_2 - 10x_3 = 13 \tag{3.24}$$

SOLUTION

Although it makes no difference which variable is eliminated first, let's eliminate x_2. If Eqs. (3.22) and (3.23) are added, the resultant Eq. (3.25) is stated in terms of x_1 and x_3:

$$\begin{array}{r} x_1 + x_2 + x_3 = 6 \\ \underline{2x_1 - x_2 + 3x_3 = 4} \\ 3x_1 \qquad + 4x_3 = 10 \end{array} \tag{3.25}$$

EXAMPLE 21 **(No Solution)** Determine the solution set for the following system of equations:

$$-2x_1 + x_2 + 3x_3 = 12 \quad (3.28)$$

$$x_1 + 2x_2 + 5x_3 = 10 \quad (3.29)$$

$$6x_1 - 3x_2 - 9x_3 = 24 \quad (3.30)$$

SOLUTION

Variable x_1 can be eliminated by multiplying Eq. (3.29) by +2 and adding it to Eq. (3.28), as follows:

$$\begin{array}{rr} & -2x_1 + x_2 + 3x_3 = 12 \\ [2 \cdot \text{Eq. (3.29)}] \rightarrow & \underline{2x_1 + 4x_2 + 10x_3 = 20} \\ & 5x_2 + 13x_3 = 32 \end{array} \quad (3.31)$$

Similarly, x_1 can be eliminated by multiplying Eq. (3.29) by −6 and adding the resulting equation to Eq. (3.30), or

$$\begin{array}{rr} [-6 \cdot \text{Eq. (3.29)}] \rightarrow & -6x_1 - 12x_2 - 30x_3 = -60 \\ & \underline{6x_1 - 3x_2 - 9x_3 = 24} \\ & -15x_2 - 39x_3 = -36 \end{array} \quad (3.32)$$

Eliminating x_1 leaves the (2 × 2) system

$$5x_2 + 13x_3 = 32 \quad (3.31)$$

$$-15x_2 - 39x_3 = -36 \quad (3.32)$$

To eliminate x_2, Eq. (3.31) is multiplied by +3 and added to Eq. (3.32), or

$$\begin{array}{rr} [3 \cdot \text{Eq. (3.31)}] \rightarrow & 15x_2 + 39x_3 = 96 \\ & \underline{-15x_2 - 39x_3 = -36} \\ & 0 = 60 \end{array} \quad (3.33)$$

Note that Eq. (3.33) is a contradiction, meaning that there is no solution to the original system of equations.

EXAMPLE 22 **(Infinitely Many Solutions)** Determine the solution set for the system of equations

$$x_1 + x_2 + x_3 = 20 \tag{3.34}$$

$$2x_1 - 3x_2 + x_3 = -5 \tag{3.35}$$

$$6x_1 - 4x_2 + 4x_3 = 30 \tag{3.36}$$

SOLUTION

Verify that x_3 can be eliminated and Eq. (3.37) can be found by multiplying Eq. (3.34) by -1 and adding this new equation to Eq. (3.35):

$$x_1 - 4x_2 = -25 \tag{3.37}$$

Also verify that Eq. (3.38) is formed by multiplying Eq. (3.34) by -4 and adding this to Eq. (3.36):

$$2x_1 - 8x_2 = -50 \tag{3.38}$$

To eliminate x_1 from Eqs. (3.37) and (3.38), Eq. (3.37) may be multiplied by -2 and added to Eq. (3.38). When these operations are performed, Eq. (3.39) is an identity:

$$\begin{array}{r} -2x_1 + 8x_2 = 50 \\ \underline{2x_1 - 8x_2 = -50} \\ 0 = 0 \end{array} \tag{3.39}$$

This is the signal that there are infinitely many solutions to the original system.

To determine particular members of the solution set, return to one of the last meaningful equations generated during the elimination procedure [Eqs. (3.37) and (3.38)]. Then, solve for one of the variables in terms of the other. To illustrate, let's solve Eq. (3.37) for x_1.

$$x_1 = 4x_2 - 25 \tag{3.40}$$

Now, we substitute the right side of this equation into one of the original equations wherever x_1 appears. If we substitute into Eq. (3.34), we get

$$(4x_2 - 25) + x_2 + x_3 = 20$$

$$5x_2 + x_3 = 45$$

Solving for x_3 gives

$$x_3 = 45 - 5x_2 \tag{3.41}$$

Equations (3.40) and (3.41) state x_1 and x_2 in terms of x_3. Thus, one way in which we can specify the solution set is

$$\boxed{\begin{array}{l} x_2 \text{ arbitrary} \\ x_1 = 4x_2 - 25 \\ x_3 = 45 - 5x_2 \end{array}}$$

Using this specification, verify that *one* solution to the original system is $x_1 = -5$, $x_2 = 5$, and $x_3 = 20$.

❑